Journal of Approximation Theory **99**, 54–67 (1999)
Article ID jath.1998.3313, available online at http://www.idealibrary.com on IDEAL®

On the Nonnegative Solution of a Freud Three-Term Recurrence

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Communicated by Walter Van Assche

Received October 22, 1997; accepted in revised form August 11, 1998

This article deals with the sequence $\xi = \{\xi_n\}_{n=0,1,\dots}$ defined by the three-term recurrence $n = 4\xi_n(\xi_{n-1} + \xi_n + \xi_{n+1})$, $n = 1, 2, \dots$, and by the initial conditions $\xi_0 = 0$, $\xi_1 = \Gamma(3/4)/\Gamma(1/4)$. Owing both to connections between the ξ_n's and orthonormal polynomials with respect to the weight function w: $w(x) = \exp(-x^4)$ and to difficulties that arise when one attempts to compute its elements, the sequence ξ has been studied by many authors. Properties of ξ have been shown and computational algorithms provided. In this paper we show further properties of ξ. First we establish bounds for the departure of ξ from the sequence to which it asymptotically converges. Then we prove that ξ is an increasing sequence.

Key Words: Freud recurrence; orthonormal polynomials.

1. INTRODUCTION

We consider the three-term recurrence

$$n = 4\xi_n(\xi_{n-1} + \xi_n + \xi_{n+1}), \qquad n = 1, 2, \dots, \tag{1.1}$$

and study the solution $\xi = \{\xi_n\}_{n=0,1,\dots}$ satisfying the initial conditions

$$\xi_0 = 0, \qquad \xi_1 = \Gamma(3/4)/\Gamma(1/4). \tag{1.2}$$

Our interest in this matter can be motivated as follows.

First of all we point out that the recurrence in (1.1) and its solution ξ defined by (1.1) and (1.2) are closely related to the most investigated Freud system of polynomials.

* The work of the second author has been performed within the activities of the National Research Project "Analisi Numerica e Matematica Computazionale" and supported by the related M.U.R.S.T. funds.

0021-9045/99 $30.00

The Freud systems of polynomials have been introduced in [1], and subsequently investigated in many papers (see, e.g., [2–4, 7, 9, 11, 13] and [12, Sect. 4], where also extended references can be found), to approximate functions in weighted L^p spaces on the real line. They are orthonormal with respect to even weight functions of the type

$$w\colon w(x) = |x|^\rho \exp(-|x|^\beta), \qquad \rho > -1, \quad \beta > 0. \tag{1.3}$$

We focus on the system of orthonormal polynomials $\{p_n\}_{n=0,1,\ldots}$, $p_n(x) = \gamma_n x^n +$ lower terms, which is determined by the weight in (1.3) and by assuming $\gamma_n > 0$, $n = 0, 1, \ldots$, and we denote by

$$\begin{gathered} x p_n(x) = a_{n+1} p_{n+1}(x) + a_n p_{n-1}(x), \qquad n = 0, 1, \ldots, \\ a_0 = 0, \qquad a_n = \gamma_{n-1}/\gamma_n, \quad n > 0, \end{gathered} \tag{1.4}$$

the three-term recurrence that defines such a system $\{p_n\}_{n=0,1,\ldots}$.

It has been proved by Freud in [3] that, in the case of $\beta = 4$, the sequence $\xi = \{\xi_n\}_{n=0,1,\ldots}$, $\xi_n = a_n^2$, is the solution of

$$n + \frac{\rho}{2}[1 - (-1)^n] = 4\xi_n(\xi_{n-1} + \xi_n + \xi_{n+1}), \qquad n = 1, 2, \ldots, \tag{1.5}$$

$$\xi_0 = 0, \qquad \xi_1 = \Gamma((\rho+3)/4)/\Gamma((\rho+1)/4). \tag{1.6}$$

Thus the solution ξ gives the coefficients of the three-term recurrence (1.4). This is of great interest since the a_n's play a central role in many questions. For instance, the greatest zero of p_n can be expressed in terms of a_n (see, e.g., [4, 8, 12]), so that any information about the asymptotic behavior of the solution ξ can be used to get estimates of the greatest zeros of the p_n's. Another interesting application is shown in [5]. Here the a_n's are used to study electrostatic interacting particle models associated with Freud weights. Special attention is given to the case $\rho = 0$, $\beta = 4$, showing that it is possible to express the equilibrium energy of the model entirely in terms of the a_n's given by (1.1) and (1.2).

A second reason of interest in the subject is that the solution ξ defined by (1.5) and (1.6) is the unique nonnegative solution of the recurrence in (1.5) no matter what $\rho > -1$ is (see [6, 10]). This adds interest to the matter as it implies that the computational problem and the algorithm one naturally derives from (1.5) and (1.6) will necessarily be, respectively, ill-conditioned and unstable. Effective methods and stable algorithms to compute subsequences of ξ can be found in [6].

In this paper we consider the particular case of $\rho = 0$, $\beta = 4$, that leads to the recurrence (1.1) and to the sequence ξ defined by (1.1) and (1.2).

The outline of the article is as follows. In Section 2 we briefly introduce the results from [6] we will use in the sequel, along with the basic notation and definitions we require. In Section 3 some preliminary results are proved. Finally, Section 4 deals with our main results. First (Theorem 4.1) we give an estimate of the departure of ξ from the sequence

$$\eta\colon \eta_n = \frac{\sqrt{n}}{2\sqrt{3}}, \qquad n = 0, 1, ..., \tag{1.7}$$

which has been proved in [3] enjoying the property

$$\lim_{n\to\infty} \frac{\eta_n}{\xi_n} = 1. \tag{1.8}$$

Then (Theorem 4.2) we use this estimate to prove that the solution ξ defined by (1.1) and (1.2) is an increasing sequence.

2. PRELIMINARIES

The basic notation that will be used in this paper is as follows:

R the set of all real numbers.

$\mathbf{R}^n$ the n-dimensional real linear space.

N the set of nonnegative integers.

$\mathbf{N}^+$ the set of positive integers.

$F_n\colon \mathbf{R}^2 \to \mathbf{R}$, $F_n(t_1, t_2) = \frac{1}{2}(-t_1 - t_2 + \sqrt{(t_1+t_2)^2 + n})$, $n \in \mathbf{N}^+$.

$G_n\colon \mathbf{R} \to \mathbf{R}$, $G_n(t) = \frac{1}{2}(-2t + \sqrt{4t^2 + n})$, $n \in \mathbf{N}$.

X the linear space of the sequences $x = \{x_i\}_{i \in \mathbf{N}}$, $y = \{y_i\}_{i \in \mathbf{N}}$, etc.

$T\colon \mathbf{X} \to \mathbf{X}$; $T(x)\colon (T(x))_0 = 0$, $(T(x))_n = F_n(x_{n-1}, x_{n+1})$, $n \in \mathbf{N}^+$.

$S\colon \mathbf{X} \to \mathbf{X}$; $S(x)\colon (S(x))_n = G_n(x_n)$, $n \in \mathbf{N}$.

Also, we need some definitions.
Let $\xi^{(k)}$, $\eta^{(k)}$, $k \in \mathbf{N}$, be the sequences defined by

$$\xi^{(0)} = 0, \qquad \xi^{(k)} = T(\xi^{(k-1)}), \quad k \in \mathbf{N}^+, \tag{2.1}$$

$$\eta^{(0)} = 0, \qquad \eta^{(k)} = S(\eta^{(k-1)}), \quad k \in \mathbf{N}^+. \tag{2.2}$$

One has

$$\xi^{(1)}\colon \xi_n^{(1)} = \frac{1}{2}\sqrt{n}, \qquad n \in \mathbf{N}, \tag{2.3}$$

$$\eta^{(1)} = \xi^{(1)}, \qquad \eta^{(2)}\colon \eta_n^{(2)} = \frac{\sqrt{2}-1}{2}\sqrt{n}, \quad n \in \mathbf{N}. \tag{2.4}$$

The operator T, as well as the sequences $\xi^{(k)}$, $k \in \mathbf{N}$, are the same as the operator T and the sequences $\xi^{(k)}$, $k \in \mathbf{N}$, considered by Lew and Quarles [6]. In [6] it has been also proved that one has

$$\xi^{(0)} < \xi^{(2)} < \cdots < \xi^{(2k-2)} < \xi^{(2k)} < \cdots < \xi < \cdots$$
$$< \xi^{(2k+1)} < \xi^{(2k-1)} < \cdots < \xi^{(3)} < \xi^{(1)}, \tag{2.5}$$

and

$$\lim_{k \to \infty} \xi_n^{(k)} = \xi_n, \qquad n \in \mathbf{N}^+. \tag{2.6}$$

The results in (2.5) and (2.6), in addition to the one in (1.8), play an important role through the article.

Another result in [6] that will be mentioned later (see Remark 4.2) is an asymptotic series for ξ_n. We write it in the form

$$(12/n)^{1/2}\, \xi_n = 1 + \frac{1}{24n^2} - \frac{7}{576n^4} + \cdots. \tag{2.7}$$

3. SOME LEMMAS

In this section we prove some lemmas. Lemma 3.1 is used to prove that the sequences $\xi^{(k)}$, $k \in \mathbf{N}^+$, in (2.1) are contained in the region $\mathscr{R}$ defined in (3.1). Lemma 3.6 shows that also the sequences $\eta^{(k)}$, $k \in \mathbf{N}^+$, in (2.2) belong to the inclusion region $\mathscr{R}$. The remaining lemmas deal with sequences x and y in $\mathscr{R}$ and their images $T(x)$, $T(y)$ and $S(x)$, giving bounds for the difference sequences $T(x) - T(y)$ and $T(x) - S(x)$.

The scheme is as follows. Lemma 3.1 is used in Lemma 3.4. Lemma 3.2 is used in Lemma 3.3. Lemma 3.5 is used in Lemma 3.6. Finally, Lemmas 3.3, 3.4, 3.5, and 3.6 are used to prove Theorem 4.1 in Section 4.

A straightforward computation leads to the following lemma.

LEMMA 3.1. *Let* x: $x_n = c\sqrt{n}$, $n \in \mathbf{N}$, *where* c *is a positive constant. One has*

$$(T(x))_n > (S(x))_n, \qquad n \in \mathbf{N}^+.$$

Now observe that from (2.1)–(2.4) and Lemma 3.1 it follows

$$\xi^{(2)} > \eta^{(2)}.$$

Thus, using (2.3) again and the result in (2.5), we can assert that the sequences $\xi^{(k)}$, $k \in \mathbf{N}^+$, satisfy the restriction

$$\frac{\sqrt{2}-1}{2}\sqrt{n} \leqslant \xi_n^{(k)} \leqslant \frac{1}{2}\sqrt{n}, \qquad n \in \mathbf{N}, \quad k \in \mathbf{N}^+,$$

so that they are contained in the region

$$\mathscr{R} = \left\{ (t, u) \in \mathbf{R}^2\colon 0 \leqslant t < +\infty, \frac{\sqrt{2}-1}{2}\sqrt{t} \leqslant u \leqslant \frac{1}{2}\sqrt{t} \right\}. \tag{3.1}$$

LEMMA 3.2. *Let x, y be two sequences contained in the region $\mathscr{R}$. Moreover, let*

$$|x_n - y_n| \leqslant \varepsilon_n, \qquad n \in \mathbf{N}^+.$$

Then, for each $n \geqslant 3$, one has

$$|(T(x))_n - (T(y))_n| \leqslant C_1(\varepsilon_{n-1} + \varepsilon_{n+1}), \tag{3.2}$$

where

$$C_1 = 0.311017763\ldots. \tag{3.3}$$

Proof. Since, for any $x \in \mathbf{X}$, $(T(x))_n$ depends only on $x_{n-1} + x_{n+1}$, we can write

$$(T(x))_n - (T(y))_n = u_n(a_n) - u_n(b_n), \qquad n \in \mathbf{N}^+,$$

where $u_n\colon \mathbf{R} \to \mathbf{R}$ is the function defined by

$$u_n(\sigma) = \tfrac{1}{2}\left(\sqrt{\sigma^2 + n} - \sigma\right), \tag{3.4}$$

and $a_n = x_{n-1} + x_{n+1}$, $b_n = y_{n-1} + y_{n+1}$.

We shall prove the bound in (3.2) by using the Lagrange formula.

First we consider the derivative u_n' of the function in (3.4). A standard argument proves that u_n' is a negative increasing function in $\mathbf{R}$. Since both x and y are assumed to belong to the inclusion region $\mathscr{R}$, we have

$$a_n, b_n \geqslant z_n := \frac{\sqrt{2}-1}{2}\left(\sqrt{n-1} + \sqrt{n+1}\right)$$

and, consequently,

$$\begin{aligned} |(T(x))_n - (T(y))_n| &= |u_n(a_n) - u_n(b_n)| \leqslant |u_n'(z_n)|\ |a_n - b_n| \\ &= |u_n'(z_n)|\ |x_{n-1} + x_{n+1} - y_{n-1} - y_{n+1}| \\ &\leqslant |u_n'(z_n)|\ (\varepsilon_{n-1} + \varepsilon_{n+1}), \qquad n \in \mathbf{N}^+. \end{aligned}$$

Second, we replace $|u_n'(z_n)|$ with $\max_{n\geqslant 3}|u_n'(z_n)|$, which is equal to $|u_3'(z_3)|$ since the sequence $\{u_n'(z_n)\}$ is easily seen to be a negative and increasing sequence. Now the proof can be concluded by observing that $|u_3'(z_3)|$ is equal to the constant C_1 in (3.3). ▮

LEMMA 3.3. *Let x, y be two sequences contained in the region $\mathscr{R}$. Let*

$$|x_n - y_n| \leqslant \varepsilon_n, \qquad n \in \mathbf{N}^+,$$

and

$$\varepsilon_n = \frac{K}{n\sqrt{n}}, \tag{3.5}$$

where K is a positive constant.

Then, for each $n \geqslant 3$, one has

$$|(T(x))_n - (T(y))_n| \leqslant C_2 \frac{K}{n\sqrt{n}}, \tag{3.6}$$

where

$$C_2 = 0.773388078... . \tag{3.7}$$

Proof. The hypotheses of this lemma imply those of Lemma 3.2. So we can use (3.2) and write, taking (3.5) into account,

$$\begin{aligned}
&|(T(x))_n - (T(y))_n| \\
&\quad \leqslant C_1 K \left(\frac{1}{(n-1)\sqrt{n-1}} + \frac{1}{(n+1)\sqrt{n+1}} \right) \\
&\quad = \frac{C_1 K}{n\sqrt{n}} \left(n\sqrt{n} \left(\frac{1}{(n-1)\sqrt{n-1}} + \frac{1}{(n+1)\sqrt{n+1}} \right) \right).
\end{aligned} \tag{3.8}$$

The function

$$v\colon v(t) = t\sqrt{t} \left(\frac{1}{(t-1)\sqrt{t-1}} + \frac{1}{(t+1)\sqrt{t+1}} \right)$$

is a positive and decreasing function in $(1, +\infty)$. Indeed, standard computations show that v' is negative in $(1, +\infty)$. Consequently, we have

$$n\sqrt{n} \left(\frac{1}{(n-1)\sqrt{n-1}} + \frac{1}{(n+1)\sqrt{n+1}} \right) \leqslant v(3), \qquad n \geqslant 3, \tag{3.9}$$

and the assertion in (3.6) follows from (3.8) and (3.9) by observing that $C_1 v(3)$ is equal to the constant C_2 in (3.7). ▮

Remark 3.1. In the previous two lemmas we restricted ourselves to prove inequalities (3.2) and (3.6) for $n \geqslant 3$. In fact, considering $n \geqslant 2$ in Lemma 3.3 would have led to a bound (3.6) with a constant $C_2 > 1$. Actually, a constant C_2 greater than 1 does not fit the sequel (see Theorem 4.1).

As Lemma 3.1, the following lemma deals with the applications of both the operators T and S to a sequence x of the type

$$x\colon x_n = c\sqrt{n}, \qquad n \in \mathbf{N},$$

but now the restriction $x \in \mathscr{R}$ is assumed.

LEMMA 3.4. *Let*

$$x\colon x_n = c\sqrt{n}, \qquad n \in \mathbf{N}, \qquad c \in \left[\frac{\sqrt{2}-1}{2}, \frac{1}{2}\right]. \tag{3.10}$$

For each $n \geqslant 3$ one has

$$0 < (T(x))_n - (S(x))_n \leqslant \frac{C_3}{n\sqrt{n}}, \tag{3.11}$$

where

$$C_3 = 0.019990444\ldots. \tag{3.12}$$

Proof. We divide the proof in four steps.

(I) We observe that from Lemma 3.1 it follows

$$0 < (T(x))_n - (S(x))_n, \qquad n \in \mathbf{N}^+. \tag{3.13}$$

(II) We prove that

$$(T(x))_n - (S(x))_n < \tfrac{1}{2}\tau_n(c)\,\Delta_n, \qquad n \in \mathbf{N}^+, \tag{3.14}$$

where

$$\Delta_n = 2\sqrt{n} - \sqrt{n-1} - \sqrt{n+1}$$

and

$$\tau_n(c) = c - \frac{c^2}{\sqrt{4c^2+1}}\,\frac{\sqrt{n-1}+\sqrt{n+1}}{\sqrt{n}}.$$

Proof. Simple calculations show that

$$(T(x))_n - (S(x))_n$$
$$= -\tfrac{1}{2}[c\Delta_n - (\sqrt{n + c^2(2\sqrt{n})^2} - \sqrt{n + c^2(\sqrt{n-1} + \sqrt{n+1})^2})].$$

Now we apply the Lagrange formula getting

$$(T(x))_n - (S(x))_n = \frac{1}{2}\left[c - \frac{c^2(\sqrt{n-1} + \sqrt{n+1} + \vartheta\Delta_n)}{\sqrt{n + c^2(\sqrt{n-1} + \sqrt{n+1} + \vartheta\Delta_n)^2}}\right]\Delta_n,$$
$$0 < \vartheta < 1. \tag{3.15}$$

Then we observe that Δ_n is greater than zero and we obtain an upper-bound for $(T(x))_n - (S(x))_n$ by taking $\vartheta = 0$ in the numerator of the ratio in (3.15) and $\vartheta = 1$ in the denominator. This leads to (3.14).

(III) We prove that

$$\Delta_n < \frac{0.26}{n\sqrt{n}}, \qquad n \geqslant 3. \tag{3.16}$$

Proof. This bound has been suggested by the limit relation

$$\lim_{t \to +\infty} (2\sqrt{t} - \sqrt{t-1} - \sqrt{t+1})\, t^{3/2} = 0.25. \tag{3.17}$$

First we prove that (3.16) is equivalent to the condition

$$\frac{8.45}{n^2} - \frac{0.8788}{n^4} + \frac{0.028561}{n^6} < 1, \qquad n \geqslant 3.$$

Then we observe that for $n \geqslant 3$ one has

$$\frac{8.45}{n^2} - \frac{0.8788}{n^4} + \frac{0.028561}{n^6} < \frac{8.45}{n^2} + \frac{0.028561}{n^6} \leqslant \frac{8.45}{3^2} + \frac{0.028561}{3^6} < 1.$$

This concludes the proof.

(IV) We prove that

$$\tau_n(c) < 0.15377265... . \tag{3.18}$$

Proof. First we observe that, for any c, $\tau_n(c)$ is a decreasing sequence. Thus, we can write

$$\tau_n(c) \leqslant \tau_3(c) = c - \frac{\sqrt{2}+2}{\sqrt{3}} \frac{c^2}{\sqrt{4c^2+1}}, \qquad n \geqslant 3, \qquad c \in \left[\frac{\sqrt{2}-1}{2}, \frac{1}{2}\right].$$

Then we study τ_3' and we find that τ_3' decreases in $[(\sqrt{2}-1)/2, 1/2]$ and vanishes at a point $c^* \in ((\sqrt{2}-1)/2, 1/2)$. As a consequence

$$\tau_3(c) \leqslant \tau_3(c^*), \qquad n \geqslant 3, \qquad c \in \left[\frac{\sqrt{2}-1}{2}, \frac{1}{2}\right].$$

Finally, we use the fixed-point method to find c^*, which is given by

$$c^* = 0.411905191\ldots,$$

and we compute $\tau_3(c^*)$ obtaining (3.18).

Now we are in a position to conclude the proof. In fact, (3.13) proves the left-hand side inequality in (3.11), whereas the right-hand side inequality in (3.11) and the value of C_3 in (3.12) follow from (3.14), (3.16), and (3.18). ∎

We conclude this section with a few remarks and two lemmas related to the function G_n defined in Section 2 and to the operator S.

The function G_n is a simplified version of F_n. It has been defined taking into account that, actually, F_n is a function of $t_1 + t_2$. One has

$$G_n\left(\frac{t_1+t_2}{2}\right) = F_n(t_1, t_2).$$

In addition, function G_n leads to the operator S, which is such that

$$\eta = S(\eta), \tag{3.19}$$

where $\eta \in \mathscr{R}$ is the sequence defined in (1.7) and enjoying property (1.8).

The following lemma shows that S is a contractive map in the inclusion region $\mathscr{R}$ and that the sequences $\eta^{(k)}$ it generates according to (2.2) converge to η.

LEMMA 3.5. *One has*

$$\lim_{k\to\infty} \eta_n^{(k)} = \eta_n, \qquad n \in \mathbf{N}.$$

Proof. First we prove that S is a contractive map in $\mathscr{R}$, namely that one has

$$|S(x) - S(y)| < L\Delta_{xy}, \qquad L < 1,$$

for any couple of sequences $x, y \in \mathscr{R}$ satisfying the restriction

$$|x - y| < \Delta_{xy}.$$

To do so, we observe that the derivative G_n' of the function G_n is a negative and increasing function in $\mathbf{R}$. This follows by noting that $G_n(t) = u_n(2t)$ (see (3.4) for the definition of the function u_n) and recalling the behavior of u_n' observed in the proof of Lemma 3.2.

Consequently, one has

$$\max_{t \in [((\sqrt{2}-1)/2)\sqrt{n},\,(1/2)\sqrt{n}]} |G_n'(t)| = \left| G_n'\left(\frac{\sqrt{2}-1}{2}\sqrt{n}\right)\right|$$

and an easy computation shows that $|G_n'(((\sqrt{2}-1)/2)\sqrt{n})|$ does not depend on n and is a real number strictly less than 1. So we can set L equal to this number, i.e.: $L = |G_n'(((\sqrt{2}-1)/2)\sqrt{n})| = 0.617316567...$.

Then the proof can be completed by a standard argument by observing that it follows from (2.4) that $\eta^{(1)}$ and $\eta^{(2)}$ lie on the boundary of $\mathcal{R}$ (see (3.1)). ∎

Lemma 3.6. *All sequences $\eta^{(k)}$, $k \geqslant 1$, are of the type*

$$\eta^{(k)} = c_k \sqrt{n}, \tag{3.20}$$

where

$$c_k \in \left[\frac{\sqrt{2}-1}{2}, \frac{1}{2}\right]. \tag{3.21}$$

Proof. The assertion in (3.20) can be easily proved by induction using the definitions of G_n, of S and of the sequences $\eta^{(k)}$. Then, taking into account (2.3) and (2.4), (3.21) immediately follows from (3.20) and Lemma 3.5. ∎

4. MAIN RESULTS

The following theorem gives an estimate of the departure of the solution ξ from the sequence η in (1.7). It invokes Lemma 3.3, 3.4, 3.5, and 3.6.

Theorem 4.1. *The sequence ξ defined by* (1.1) *and* (1.2) *and the sequence η defined by* (1.7) *satisfy the condition*

$$|\xi_n - \eta_n| \leqslant \frac{C}{n\sqrt{n}}, \qquad n \geqslant 3, \tag{4.1}$$

where

$$C = 0.08821444... . \tag{4.2}$$

Proof. Consider the sequences

$$e^{(k)}\colon e_n^{(k)} = |\xi_n^{(k)} - \eta_n^{(k)}|, \qquad n \in \mathbf{N}, \quad k \in \mathbf{N}^+.$$

From (2.4) it follows $e^{(1)} = 0$.
For $k > 1$, we can write

$$\begin{aligned} e_n^{(k)} &= |(T(\xi^{(k-1)}))_n - (S(\eta^{(k-1)}))_n| \\ &\leqslant |(T(\xi^{(k-1)}))_n - (T(\eta^{(k-1)}))_n| + |(T(\eta^{(k-1)}))_n - (S(\eta^{(k-1)}))_n|. \end{aligned} \tag{4.3}$$

For $k = 2$, we can use (2.4) again to get $|(T(\xi^{(1)}))_n - (T(\eta^{(1)}))_n| = 0$. Also, we observe that, by virtue of Lemma 3.6, $\eta^{(1)}$ is of the type (3.10), so that we can apply Lemma 3.4 to get a bound for $|(T(\eta^{(k-1)}))_n - (S(\eta^{(k-1)}))_n|$. In this way it follows from (4.3),

$$e_n^{(2)} \leqslant 0 + \frac{C_3}{n\sqrt{n}}, \qquad n \geqslant 3. \tag{4.4}$$

All steps related to $k \geqslant 3$ require a same argument. They need also Lemma 3.3.

Take $k = 3$. Lemma 3.3 can be applied by virtue of (4.4). Again we can invoke Lemma 3.6 to apply Lemma 3.4. From (4.3) we get

$$e_n^{(3)} \leqslant \frac{C_2 C_3}{n\sqrt{n}} + \frac{C_3}{n\sqrt{n}} = C_3 \frac{1 + C_2}{n\sqrt{n}}, \qquad n \geqslant 3. \tag{4.5}$$

Repeating iteratively the same argument for $k > 3$, we obtain the inequalities

$$e_n^{(k)} \leqslant C_3 \frac{1 + C_2 + C_2^2 + \cdots + C_2^{k-2}}{n\sqrt{n}}, \qquad n \geqslant 3,$$

that for $k = 2$ and 3 reduce to (4.4) and (4.5).

Finally, since C_2 is less than 1 (see (3.7)), we can bound $e_n^{(k)}$ uniformly with respect to k and write

$$\begin{aligned} e_n^{(k)} = |\xi_n^{(k)} - \eta_n^{(k)}| &\leqslant \frac{C_3}{n\sqrt{n}} \frac{1 - C_2^{k-1}}{1 - C_2} \\ &\leqslant \frac{C_3}{1 - C_2} \frac{1}{n\sqrt{n}}, \qquad k \in \mathbf{N}^+, \quad n \geqslant 3. \end{aligned}$$

As k tends to infinity, from (2.6) and Lemma 3.5, we get

$$|\xi_n - \eta_n| \leqslant \frac{C_3}{1-C_2} \frac{1}{n\sqrt{n}}, \qquad n \geqslant 3,$$

which proves (4.1). Indeed, one has that $C_3/(1-C_2)$ is equal to the constant C in (4.2). ∎

Remark 4.1. Even if we were not able to prove this, inequality (4.1) actually holds also for $n = 1, 2$. As a matter of fact,

$$\begin{aligned} \xi_1 - \eta_1 &= \Gamma(3/4)/\Gamma(1/4) - 1/(2\sqrt{3}) = 0.045104756... , \\ \xi_2 - \eta_2 &= 1/(4\xi_1) - \xi_1 - \sqrt{2}/(2\sqrt{3}) = -0.01857892.../(2\sqrt{2}). \end{aligned} \tag{4.6}$$

However, since (4.6) is obtained from ξ_1 via (1.1), and this is the first step of a strongly unstable algorithm, in the following theorem we shall use inequality (4.1) only for $n \geqslant 3$.

Remark 4.2. The asymptotic series for ξ_n in (2.7) shows that

$$\lim_{n\to\infty} n\sqrt{n}\, |\xi_n - \eta_n| = \frac{1}{48\sqrt{3}}.$$

As a consequence, the infinitesimal order in (4.1) can not be improved.

THEOREM 4.2. *The sequence ξ is increasing.*

Proof. First we prove that

$$\xi_n < \xi_{n+1}, \qquad n \geqslant 3, \tag{4.7}$$

by using Theorem 4.1.

To do so, we write (4.1) in the form

$$\frac{\sqrt{n}}{2\sqrt{3}} - \frac{C}{n\sqrt{n}} \leqslant \xi_n \leqslant \frac{\sqrt{n}}{2\sqrt{3}} + \frac{C}{n\sqrt{n}}, \qquad n \geqslant 3,$$

and then we observe that inequalities (4.7) are certainly verified if

$$\frac{\sqrt{n}}{2\sqrt{3}} + \frac{C}{n\sqrt{n}} < \frac{\sqrt{n+1}}{2\sqrt{3}} - \frac{C}{(n+1)\sqrt{n+1}}, \qquad n \geqslant 3. \tag{4.8}$$

These inequalities are easily seen to be equivalent to the conditions

$$\frac{1}{2\sqrt{3}\,C} > \frac{1}{n}\left(1+\sqrt{\frac{n+1}{n}}\right) + \frac{1}{n+1}\left(1+\sqrt{\frac{n}{n+1}}\right), \qquad n \geqslant 3. \tag{4.9}$$

Since

$$\frac{1}{n+1}\left(1+\sqrt{\frac{n}{n+1}}\right) \qquad \text{and} \qquad 1+\sqrt{\frac{n+1}{n}}$$

are decreasing sequences and (4.9) is satisfied for $n=3$, inequalities (4.7) are proved.

To complete the proof we need only to show that

$$\xi_1 < \xi_2 < \xi_3. \tag{4.10}$$

The values of ξ_1, ξ_2 and ξ_3 have been computed with great accuracy in [6] and we could conclude the proof by referring the reader to [6, Appendix A] to verify (4.10).

However, for the sake of completeness, we prefer to reduce the proof of (4.10) to conditions on $\xi_1 = \Gamma(3/4)/\Gamma(1/4)$. To do so, we use the recurrence relation (1.1) to express ξ_2 and ξ_3 in terms of ξ_1. One has

$$\begin{aligned}
\xi_3 > \xi_2 &\Leftrightarrow \frac{1/2 - \xi_2^2 - \xi_1\xi_2}{\xi_2} > \xi_2 \Leftrightarrow \xi_2 < \frac{-\xi_1 + \sqrt{\xi_1^2+4}}{4} \\
&\Leftrightarrow \frac{1}{4\xi_1} - \xi_1 < \frac{-\xi_1 + \sqrt{\xi_1^2+4}}{4} \Leftrightarrow 8\xi_1^4 - 10\xi_1^2 + 1 < 0 \\
&\Leftrightarrow \frac{5-\sqrt{17}}{8} < \xi_1^2 < \frac{5+\sqrt{17}}{8},
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
\xi_2 > \xi_1 &\Leftrightarrow \frac{1/4 - \xi_1^2}{\xi_1} > \xi_1 \\
&\Leftrightarrow \xi_1^2 < \frac{1}{8},
\end{aligned} \tag{4.12}$$

and, since (4.11) and (4.12) are true, the proof is completed. ∎

ACKNOWLEDGMENTS

The authors are grateful to Paul Nevai for his useful suggestions and to the referees for having brought to their attention the article in [5].

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